Building on Reliability: Reliability Test Planning, Part 3

The Weibull Distribution and Allowing for Failure

By Stephen N. Luko

Q How do we use the Weibull distribution for our failure mode model, allowing for one or more failures?

A In Parts 1 and 2 of this series,[1-2] we discussed zero-failure test plans that do not depend on any distributional assumptions and similar plans that use the Weibull distribution as the model for the failure mode. In both cases, a zero-failure criteria was used. Here, we continue to use the Weibull distribution but also allow for one or more failures in the plan. Readers unfamiliar with Parts 1 and 2 of this series are encouraged to read the first two articles for basic background information, including terminology and details of the Weibull distribution.

In the plans discussed here, there is a test time, t_0, beyond which any test is considered successful when its life is greater than t_0. The time t_0 is sometimes referred to as a "boggy" value. There is also a sample size, *n*, and a number of failures, *r*, allowed by the plan. When any sample fails in less time than t_0, that unit counts as a failure in the plan. In addition, there is a confidence value C (0 < *C* < 1) being used and an assumed Weibull "slope" *β*, for the employed Weibull distribution, which is the assumed model for the failure time. For review, the Weibull cumulative distribution with its two parameters is Equation 1.

$$F(t) = 1 - e^{-(\frac{t}{\eta})^{\beta}} \quad (1)$$

This can be recast as:

$$F(t) = 1 - (1 - p/100)^{(\frac{t}{B_p})^{\beta}} \quad (2)$$

where, instead of the parameter η, a B_p life parameter has been substituted for arbitrary *p*. The B_p life is a life such that there is a probability p% of failure at that time. In test planning, one objective is to determine a lower confidence bound for the parameter η, or for a related B_p life, where (0 < p < 100) is interpreted as a percent failure. For example, the B_{10} life is a life value such that the probability of failure is 10%, making the reliability at this point 100 – 10 = 90%.

The plans in this article assume β, *C, n,* and *r* are specified and the test time, t_0, is sought. In addition, there is a target B_p life specified for which we are trying to demonstrate. When the plan is met, then we can claim a confidence 100C% that the true B_p life is at least that specified by the plan. The method uses the beta distribution (not to be confused with the Weibull shape parameter *β*) and the Weibull cdf, Equation 1 with confidence *C*, sample size *n*, and maximum number of failures *r* allowed by the plan. Start by solving the following cumulative beta distribution (cdf) for *v* (see Reference 3 for general information on the beta distribution).

$$C = \int_0^v B(r+1, n-r)dy \quad (3)$$

In Equation 3, *C* is the fixed confidence chosen, and *B*(*a,b*) is a beta density function with parameters *a* = *r* + 1 and *b* = *n* - *r*. Then *C* is equal to the beta cdf evaluated at *v*. Refer to this cdf as *G(v)*. *v* is found using the inverse beta cdf with argument *C*. This is:

$$v = G^{-1}(C) \quad (4)$$

This can be calculated easily in a program such as MS Excel. Use the Excel inverse beta function as " = BETA.INV(*C*,*r* + 1, *n* - *r*,0,1)". The program returns the required value of *v*.

When *v* is determined, it is used in the Weibull cdf function, Equation 1, as *v* = *F*(t_0) where t_0 is the resulting test time. The test time t_0 is found by inverting *v* = *F*(t_0) and also applying the Weibull alternative cdf function, Equation 2. This result is a closed form solution shown in Equation 5.

$$t_0 = B_p\left(\frac{\ln(1-v)}{\ln(1-p/100)}\right)^{1/\beta} \quad (5)$$

In Equation 5, t_0 is the required test time such that in a sample of *n* units not more than *r* failures is allowed. This uses a confidence *C* and an assumed shape parameter β. Note that *C, n,* and *r* are being incorporated in the calculation of *v* and do not appear directly in Equation 5.

EXAMPLE

A demonstration test for a certain type of safety mechanical device has a requirement of $B_{10} \geq 1000$ minutes using 90% confidence. The test engineer will use $\beta = 1.5$ as this is a standard value used for this type of device industry wide. A sample of size $n = 21$ is available, and the engineer is willing to allow $r = 1$ failure during the test. It is easy to use a spreadsheet type program with built-in beta function to determine the plan for this simple case. Table 1 is an output from MS Excel that shows the inputs, the steps, and the final result.

The value $v = 0.172935$ was determined using Equations 3 and 4 with associated $n = 21$, $r = 1$, and $C = 0.9$. Test time t_0 was determined using Equation 5. In this case, $t = 1481$ test minutes are required for each of the 21 samples. One failure is allowed. If the plan is executed and not more than one failure occurs, then the B_{10} life of 1000 minutes has been demonstrated at 90% confidence. The total test time in this case is a maximum of 31,099 minutes or 518.4 hours, approximately. In practice, one could vary the sample size, the allowable number of failures, confidence, and the assumed β to see the effect on test time. For example, if we hold confidence and $r = 1$ fixed, we can create a table (Table 2) showing the effect of changing sample size.

Readers may be interested to know that the subcommittee on reliability (E11.40) has recently developed a new standard on reliability test planning. This document is to be designated as E3291 and will be published later this year. The new standard gives many more details and testing scenarios such as those discussed in this series of articles. ■

Table 1 — Inputs, Steps, and Final Result for the Example

C	0.9
N	21
R	1
beta, β	1.5
P	10
Bp	1,000
Results, Steps	
v	0.172935
t (minutes)	1,480.9
total time max	31,098.7

Table 2 — Effect of Selected Sample Size, *n*, on Test Time, t_0 (see example).

N	t_0	Total Time
38	989.8	37,612.40
25	1,314.80	32,870.00
21	1,480.90	31,098.90
16	1,784.90	28,558.40
12	2,179.30	26,151.60
8	2,903.60	23,228.80
6	3,581.10	21,486.60

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REFERENCES

1. Luko, Stephen, and Neubauer, Dean, Building on Reliability: Reliability Test Planning, Data Points, *ASTM Standardization News*, Vol. 49, No. 1, January/February 2021, pp. 52–53.
2. Luko, Stephen, and Neubauer, Dean, Building on Reliability: Reliability Test Planning, Part 2, Data Points, *ASTM Standardization News*, Vol. 49, No. 3, May/June 2021, pp. 48–49.
3. Beta Distribution, en.wikipedia.org/wiki/Beta_distribution.

Editor's Note

It is with great sadness that we announce that Data Points editor Dean Neubauer passed away in April. Dean and Stephen Luko had recently collaborated on this series of articles and others in the past, as well as on many other projects. Dean served as editor of this column during its first nine years and again from 2019 until his passing this year. His contributions to *Standardization News* and ASTM International's committee on quality and statistics (E11) will be greatly missed. Read his *SN* obituary on page 61.