Statistical Intervals

Part 3: More on the Tolerance Interval

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In this article series, we continue to develop and review the statistical interval concept focusing on the tolerance interval. We also continue to use the example from Parts 1 and 2 demonstrating the idea of a tolerance interval to show a direct comparison.

Q: What is a tolerance interval?

A: A tolerance interval is an interval constructed using a set of sample data so as to contain a specified proportion of all future observation(s) with some stated confidence. Another way to state this is to say that at least 100*p* percent of a population will fall within the tolerance interval with some stated confidence ($0 < p < 1$). Recall that the prediction type interval is only used for forecasting several (say $k \geq 1$) future observations. Note the difference; the tolerance interval is stating that at least some proportion *p* of all future observations will be contained within the interval.

We continue to require that all samples used for this analysis were selected under the same conditions and from the same population or process and that the sample was random, or the process was in a state of statistical control. In the same manner as was used for prediction, tolerance intervals can be constructed for one-sided as well as two-sided interval cases. The two-sided case takes the form of the number pair [L, U], where we state that at least some proportion, *p*, from the future output of the process or the population will fall within this interval. For one-sided cases, we use one of the forms ($-\infty$, U] or [L, ∞). We can also use a specified distribution such as the normal or the exponential, or we can construct nonparametric tolerance intervals where we are uncertain about the distribution that may apply. In addition, we can have additional variations as, for example, when we know the value of a parameter such as a standard deviation in the normal case.

Suppose we have a random sample of *n* observations $X_1, X_2, \ldots, X_n$, and we assume the data came from a normal distribution. It is very common for practitioners to construct intervals of the form

$$\overline{X} \pm ks \quad (1)$$

Such intervals are claimed to capture or include 68, 95 and 99.7 percent of the population, respectively, which the data represent. Most often, there is no discussion of sample size or confidence in the stated result. The interval is just used as-is. What the novice may fail to understand is that this interval is only true when we use the true mean μ and standard deviation σ. We really should be using $\mu \pm k\sigma$ when we make such a claim. In practice, we never know the values of μ and σ, and so these must be estimated by the sample average and standard deviation. In sample estimation there is uncertainty in any stated interval such as the ones described above. This uncertainty arises due to the fact that we are using statistics in the interval construction and that a small sample size may have been used. Tolerance intervals were invented to remove this uncertainty.

For the normal distribution case, the two-sided tolerance interval construction is of the form $\bar{x} \pm kS$, where *k* is a function of the sample size *n*, the confidence *C* and the minimum proportion *p*, claimed to be captured by the interval. The mathematics for determining the value of *k* for unknown μ and σ is complicated and must be done using numerical analysis. Approximation formulas are available, however, and many texts use such formulas. The interested reader should consult References 1 through 3 for details. Table 1 contains such factors for selected common values of C, p and n.

Table 1 – Factors (*k*) for Constructing Tolerance Limits for a Normal Distribution

n	C	p	K
22	0.95	0.90	2.264
22	0.95	0.95	2.697
30	0.95	0.90	2.140
30	0.95	0.95	2.549
50	0.95	0.90	1.996
50	0.95	0.95	2.379
75	0.95	0.90	1.917
75	0.95	0.95	2.285
100	0.95	0.90	1.874
100	0.95	0.95	2.233

For the one-sided case where σ can be assumed known, we do have a closed form formula that only requires a table of the standard normal distribution. We illustrate this for the lower bound case. The form of the lower bound is $\bar{x} - k\sigma$. The value of k is given by:

$$k = Z_0 + \frac{Z_c}{\sqrt{n}} \tag{2}$$

For confidence level C and proportion captured p, Z_0 and Z_c are determined using the standard normal distribution where $P(Z > Z_0) = 1 - p$ and $P(Z < Z_c) = C$. For example, with $C = 0.95$ and $p = 0.99$, we find $Z_0 = 2.326$ and $Z_c = 1.64$. If a sample of $n = 12$ is used, k is determined using Equation 2 as $k = 2.80$. Then if a sample of $n = 12$ is taken we can be 95 percent confident that the population that produced the data has at least 99 percent of its future output at or above $\bar{x} - 2.80\sigma$. We turn next to an illustration where both μ and σ are unknown.

In Parts 1 and 2 of this series, we considered $n = 22$ tensile adhesion test results made on U-700 alloy specimens. The load at failure had $\bar{x} = 13.71$ and $s = 3.55$, which produced a 95 percent confidence interval for μ of $12.14 \leq \mu \leq 15.28$. We now want to determine a tolerance interval for the load at failure, which will include 90 percent of the population values with 95 percent confidence. We can use Table 1 to find k for $n = 22$, $p = 0.90$ and confidence $C = 0.95$. The value of k is 2.264. Using Equation 1, the desired tolerance interval is

$$(\bar{X} \pm ks) \Rightarrow (13.71 - (2.264)(3.55), 13.71 + (2.264)(3.55)) \Rightarrow (5.67, 21.75) \tag{3}$$

Our interpretation of this tolerance interval is that we can be 95 percent confident that at least 90 percent of the load at failure values for the U-700 alloy will lie between 5.67 and 21.75 megapascals. This tolerance interval is much wider than the 95 percent confidence interval for the mean. It is also interesting to note that as $n \to \infty$, the k value approaches the Z value corresponding to the desired level of p for the normal distribution. As an example, suppose we desire $p = 0.90$ for a two-sided tolerance interval. In this case, k approaches $Z_{1-(1-p)/2} = Z_{0.95} = 1.645$ as $n \to \infty$. In fact, as $n \to \infty$, the 100p percent prediction interval for a ($k = 1$) future value approaches the tolerance interval that contains 100p percent of the distribution. To illustrate this, in Part 2, the 95 percent prediction interval for a future value was found to be $6.16 \leq X_{23} \leq 21.26$, which is slightly smaller in width than the 95 percent tolerance interval shown in this article for $n = 22$ values.

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